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METHOD FOR FINDING MIN AND MAX VALUES OF ERROR RANGE FOR CALCULATION OF MOMENT OF INERTIA

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ABSTRACT

Modern ship design practices require knowledge of a vessel's mass Moment of Inertia (MOI) for various aspects of performance analysis. To find an accurate MOI value of an object, one needs to know the object's actual shape and density to be able to calculate the MOI through integration. Determining the exact MOI for a complete vessel, comprised of thousands of items, is not practical. Instead, engineers simplify the parts of the vessel to point objects or to standard shapes like a box or a cylinder, and calculate an approximation of the MOI. The accuracy of this approximation is dependent on the number of parts the vessel is divided into and how well the shape, orientation and density of each of the simplified items resembles the real objects. The quantification of the inaccuracy involved is seldom addressed. This paper describes a method to find the absolute error range for this simplified MOI calculation by finding the extreme values the MOI approximation can generate, and quantifies the effect that an error in MOI can have on the results of various types of performance analysis.

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Table of Contents

Table of Figures

Table of Tables

Background

The Goal of this Paper

Accurate moment of inertia (MOI) calculations of a ship can be a labor intensive task. Further, it is hard to know the amount of work that will be necessary, as there are no established methods to determine what level of confidence is appropriate, nor how to quantify the error range of the answers that the various calculation methods yield. This can lead to an over investment in resources, obtaining an MOI that is insignificantly more accurate, or an under investment of resources, resulting in an answer that is too imprecise. The goals of this paper may be summarized as follows:

- Determine the needed accuracy of an MOI calculation.
- Present an efficient way of calculating MOI to the level of accuracy needed.
- Present a method that quantifies the uncertainty of the calculated MOI.
- Present a way to improve an MOI calculation to achieve the needed level of accuracy.

What is MOI

Explaining MOI

The moment of inertia of an object about a given axis is a quantity that describes how difficult it is to change the object's angular motion about that axis. MOI therefore encompasses not only how much mass the object has overall, but how far the mass of each component comprising the object is from the axis. The farther from the axis the object's mass is, the more rotational inertia the object has and the more force is required to change its rotation rate. In one sentence, it could be said that:

Moment of inertia is a measure of an object's resistance to changes in its rotation rate.

For the purposes of this paper, MOI is mass moment of inertia for rotating objects. Moment regarding bending of a plane (second moment of inertia) is not discussed in this paper.

Mathematical Definition

The MOI depends on a reference axis, and is usually specified with two subscripts. This provides clarity during three-dimensional motion, where rotation can occur about multiple axes.

Following are the mathematical equations used to calculate the mass moment of inertia[1]:

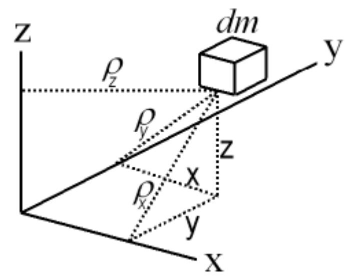


Figure 1: Calculating inertia of an object by integration

$$I_{xx} = \int \rho_x^2 dm = \int (y^2 + z^2) dm$$
$$I_{yy} = \int \rho_y^2 dm = \int (z^2 + x^2) dm$$
$$I_{zz} = \int \rho_z^2 dm = \int (x^2 + y^2) dm$$

Where: *x* is the distance from the *yz*-plane to an infinitesimal mass dm.
y is the distance from the *zx*-plane to an infinitesimal mass dm.
z is the distance from the *xy*-plane to an infinitesimal mass dm.

Radius of Gyration

In connection with MOI, the term "Radius of Gyration" is often mentioned. Radius of Gyration, also referred to as gyradius, is the radial distance from a given axis at which the mass of a body could be concentrate d without altering the rotational inertia of the body about that axis[2].

The gyradius (*k*) about a given axis can be computed in terms of the moment of inertia, and the total mass *m*;

$$k^2 = \frac{I}{m} \quad \text{or} \quad k = \sqrt{\frac{I}{m}}$$

Impact of Moment of Inertia on the Hydrodynamic Performance of a Vessel

The mass moment of inertia of a ship affects its dynamic performance in a seaway, influencing such things as accelerations, roll period, added resistance, slamming, and other important parameters. These can determine whether or not a ship can successfully execute its mission, whether the mission is carrying passengers (motion sickness incidence might be the limiting criterion), carrying cargo (cargo shifting), rescue operations (boat handling), or operations such as helicopter launch and retrieval. Also of importance are predicting design loads for the ship's structure and outfit, incidence of slamming and deck wetness, propeller emergence, or air intake into waterjet inlets.

During the various stages of design, hydrodynamicists analyze the ship using model tests and computer software to predict the motions that the ship will experience while in various seaways, at a range of ship speeds and relative wave headings. In order to do this, the MOI of the ship must be known or estimated. Generally, the gyradius is computed from the MOI and used as input to computer software or for ballasting a model prior to testing.

Especially in the very early stages of design when weight estimates lack detail, standard estimates for gyradius are used, based on historical data. There are various sources for this information, but as mentioned previously, one excellent study was presented by Cimino and Redmond[3] in 1991, giving estimates for various types of naval ships and recommendations for appropriate levels of detail for inertia calculations. They quote rules of thumb for pitch and yaw gyradius of 25% of LWL, and 35%-40% of BWL for roll gyradius. (Note that other sources may give estimates as a percentage of Length Overall (LOA) or Length Between Perpendiculars (LPP); it's important to take this into account when comparing these values.)

In order to make proper use of early stage analysis, and make correct decisions regarding the appropriate level of detail in moment of inertia calculations, one should understand the impact that gyradius has on various types of analyses.

Added Resistance in Waves

A research project by Kiss[4] demonstrated a significant impact on added resistance in waves with a variation in pitch gyradius. A model of the U.S. Coast Guard Cutter HAMILTON was tested in irregular long crested waves corresponding to sea state 5, at a range of pitch gyradii of 22.4% to 28.6% of the LPP (in this test, the model's LWL matched the LPP). As the pitch gyradius increased, the effective horsepower (EHP) also generally increased. Guidelines given by Cimino and Redmond for pitch gyradius in early design suggest a value of 24.9% of LWL, with a tolerance of +/- 0.4% of LWL. Over a range of speeds, the data from these tests show an increase in EHP of approximately 2% between the upper and lower bounds of the 0.4% tolerance (24.5% to 25.3%). While this is probably within the bounds of experimental error, the existence of a clear trend in the results implies that pitch gyradius is important when predicting the vessel's range, for example.

Seakeeping

There are a wide range of computer programs used to predict vessel motions, applying 2D and 3D analytical methods, in addition to model testing. Various statistics are generally used to represent the predicted seakeeping motions in an irregular seaway. One common response statistic used for seakeeping analysis is the "significant single-amplitude" (SSA) response value. Statistically, for a narrow-banded Gaussian-distributed motion response time history, the SSA value represents the average of the one-third highest peak values. The SSA value will be exceeded fairly often; however, it is greater than the average value and is believed to represent a characteristic or observed response value because the human observer tends to give greater visual weight to more extreme motions.

One of the more common ways to present seakeeping data is in the form of speed/heading polar diagrams. A speed/heading polar diagram shows the variation of a particular response magnitude with ship speed and relative wave heading by plotting contours of constant response magnitude for a specific sea state. Typical responses of interest include heave, pitch, roll, lateral and vertical accelerations at various locations throughout the ship, and incidence of slamming or bow wetness. A sample speed/heading polar diagram for SSA roll response in sea state 4 is shown in Figure 2.

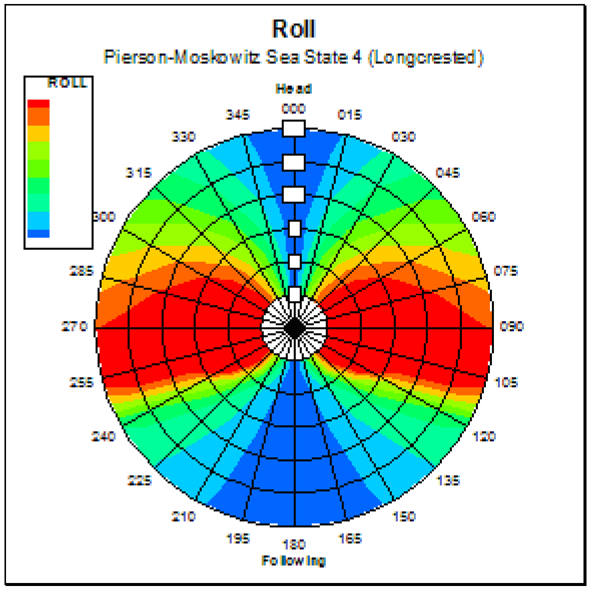


Figure 2: Polar Plot of Roll Response in a Seaway

While this type of plot is useful, it only represents a single response in a single sea state. When trying to judge the performance of the ship in a wide range of wave heights and periods, the amount of data can be overwhelming, making it difficult to rate one design against another, or to present a measure of the performance of a design in terms of its mission effectiveness. One approach to rating a ship's seakeeping performance, taking into account a variety of motion responses, is offered by the seakeeping Operability Index (OI). The OI defines the percentage of time a ship can perform a particular mission in a given sea state without exceeding any predefined motions response limits, such as a particular roll angle, heave acceleration, etc. In other words,

$$OI = \sum_{V_S} \sum_{\psi} P(V_S) \times P(\psi) \times O(V_S, \psi)$$

In this equation, $O(V_S,\psi)$ is the operability at a given speed and heading which is defined as 1 if no motion response limits are exceeded, and 0 otherwise. $P(\psi)$ is the probability of the ship operating at a particular heading. Often all headings are assumed to be equally likely. $P(V_S)$ is the probability of the ship operating at a particular speed. For a speed-independent OI, all speeds are equally likely. If a speed-time profile is available, the probability of a given ship speed can be extracted. Percent Time Operability (PTO) goes one step further, adding in the probability of the ship experiencing a range of sea conditions, usually defined by significant wave height and modal period, for a given location in the ocean.

Many seakeeping computer codes are commercially available, and many will compute PTO or OI. A ship's seakeeping operability (as measured by OI or PTO) is governed by:

- The response of the ship in waves, which is influenced by the vessel geometry and weight distribution;
- The seaway being considered;
- The operational profile (time spent at various headings and speeds);

- The limiting response criteria defined for the mission(s), which could be limiting values of roll angle, accelerations, bow slamming, incidence of motion sickness, etc.

The gyradius is taken into account when predicting the response of the ship in waves, but it is just one of the important parameters. Obviously, the ship's geometry, total weight, and center of gravity location are important as well. The gyradius that is computed by weight engineers is that of the ship in air. In water, a certain amount of fluid is entrained as the ship moves (known as “added mass”), and there are viscous damping effects from the as the vessel moves. In some cases, these added mass and damping effects are relatively small, but in other cases they can have a large influence on the motions.

The roll motions of the ship are often a limiting factor in mission effectiveness, so predicting the roll gyradius is important. The roll period of a ship in regular waves (ignoring added mass and damping) can be expressed as

$$T = \frac{2\pi k_{xx}}{\sqrt{g\overline{GM_t}}}$$

Where: T = roll period

k_{xx} = roll gyradius

g = gravitational constant

GM_t = transverse metacentric height

In tests carried out at the U.S. Naval Academy Hydromechanics Laboratory by Anderson[5], a 1:22.67 scale fully appended (skeg, bilge keels, sonar dome, shafts, struts, and propeller discs) model of the CG-58 was ballasted to match the in-water roll period of the full-scale ship, giving an in-water, or “virtual,” gyradius of 43.9% of beam. Using a Lamboley[6] test, where the model is supported and balanced on knife edges at two different heights, and then swung to determine the vertical center of gravity and the gyradius, the model was measured to have an in-air gyradius of 40.0% of beam (matching the rule of thumb given by Cimino and Redmond). The added mass and damping effects were contributing about 10% to the “virtual” roll gyradius in this case. For a vessel with heavily damped motions, such as vessel in roll with hard chines, bilge keels, or especially active stabilizers, the magnitude of error in response prediction due to estimating the roll gyradius is reduced.

To help gain an understanding of the sensitivity of various motions to gyradius, the VERES 2D strip theory-based seakeeping program[7] was used to perform a series of computer runs on a Series 60 hull (unappended), varying the pitch and roll gyradius. Roll motions in beam seas at zero speed, and pitch motions, accelerations, and motion induced sickness incidence (MSI) in head seas at 0 and 20 knots were computed in long crested irregular waves at the upper end of Sea State 4 (significant wave height of 2.5 meters, modal period range of 5 to 20 seconds), and the resulting statistical values were examined. The roll gyradii examined were 35.0%, 37.5%, and 40.0% of BWL, and the pitch gyradii were 24%, 25%, and 26% of LWL.

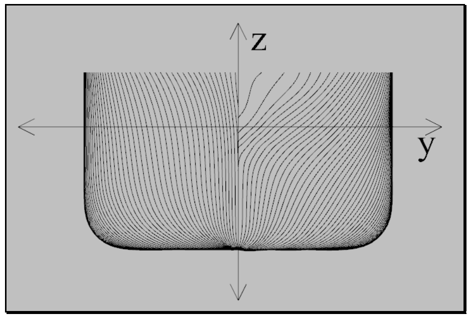


Figure 3: Series 60 Hullform Bodyplan

Length Between Perpendiculars (LBP)	121.25m
Length Waterline (LWL)	118.84m
Beam Waterline (BWL)	16.26m
Draft	6.5m
Displacement	7465 tonnes
Trim	0 degrees
KG	5m
GMt	1.645m
k_{xx}/BWL Values	35%, 37.5%, 40.0%
k_{yy}/LWL Values	24%, 25%, 26%

Table 1: Series 60 Hullform Particulars

The “virtual” roll gyradius in water, computed from the natural roll period in water, was found to be 5%-7% higher than the roll gyradius in air. This hull was unappended, so these figures were expected to be lower than the 10% difference found in the CG-58 study.

The first step in the prediction of motions in irregular waves is the measurement (with model tests) or computation (with computer software) of the Response Amplitude Operators (RAOs), essentially transfer functions for computing motions in a given irregular seaway (represented by a wave energy spectrum). An RAO plot typically shows a given motion (e.g., roll), normalized by wave height or wave slope, versus wave frequency or wave period. One interesting aspect of the RAO plot for certain motion responses is the resonant peak, showing the frequency at which response resonance occurs.

Figures 4 and 5 show the RAO plots of roll and pitch response respectively.

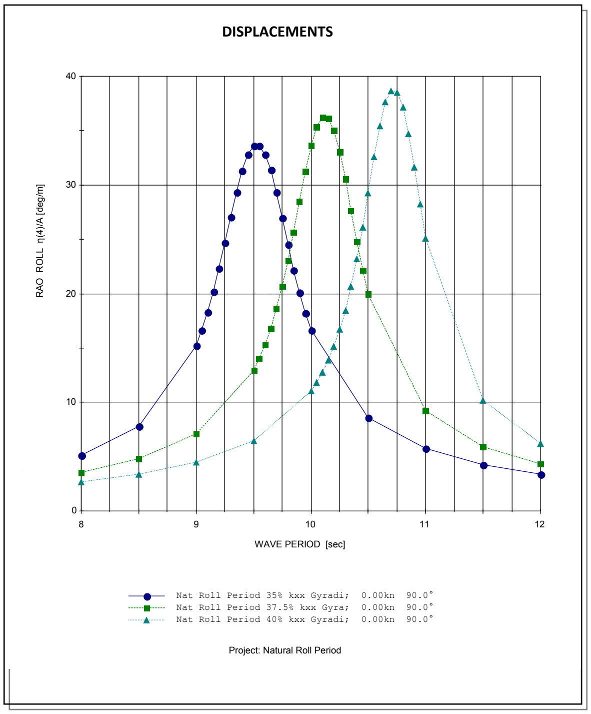


Figure 4: Roll RAO vs. Wave Period (showing only a portion of the wave period range computed)

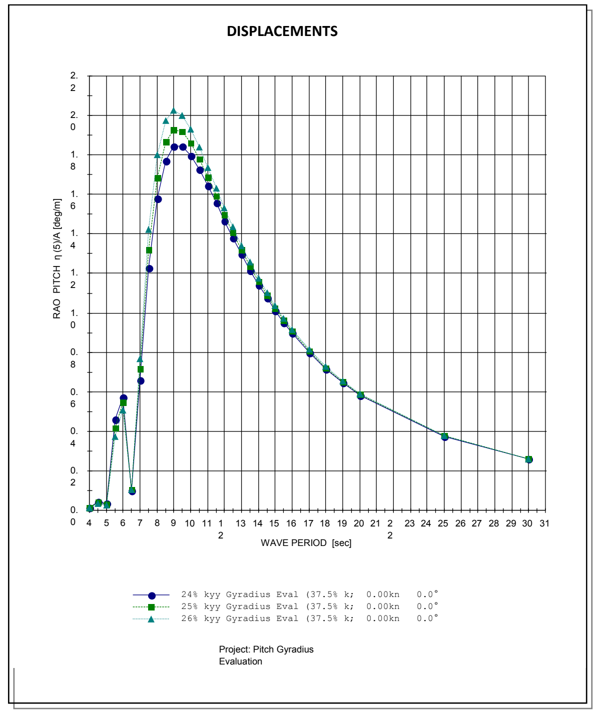


Figure 5: Pitch RAO vs. Wave Period

As expected, varying the gyradius both shifts the resonant period and changes the peak amplitude response.

Using these RAO's and a wave energy spectrum (in this case a Bretschneider spectrum, referred to in VERES as a two-parameter Pierson-Moskowitz spectrum), the principle of linear superposition can be applied in order to compute motion responses. Figures 6, 7, 8, 9, and 10 show the RMS values of roll angle, pitch angle, vertical accelerations at the bow, vertical accelerations at midships, and motion sickness incidence (MSI) at the bow. All plots show 0 and 20 knots forward speed in head seas, except for roll, which is for 0 knots in beam seas.

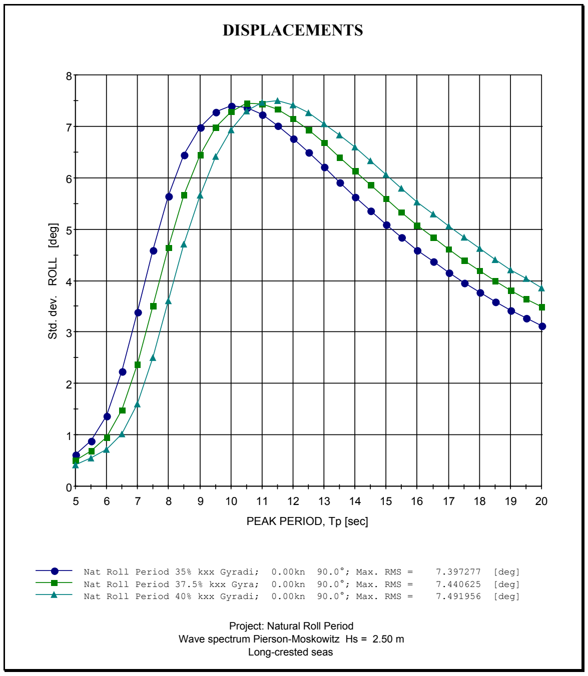


Figure 6: RMS Roll vs. Peak Period

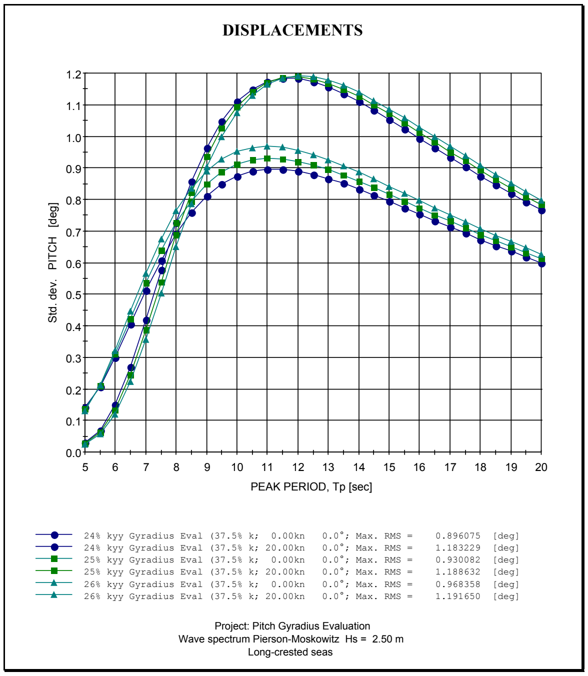


Figure 7: RMS Pitch vs. Peak Period

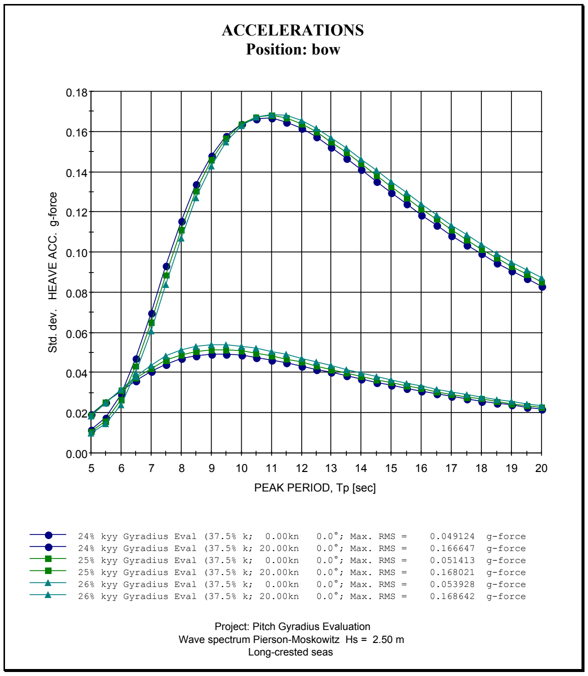


Figure 8: RMS Heave Acceleration @ Bow vs. Peak Period

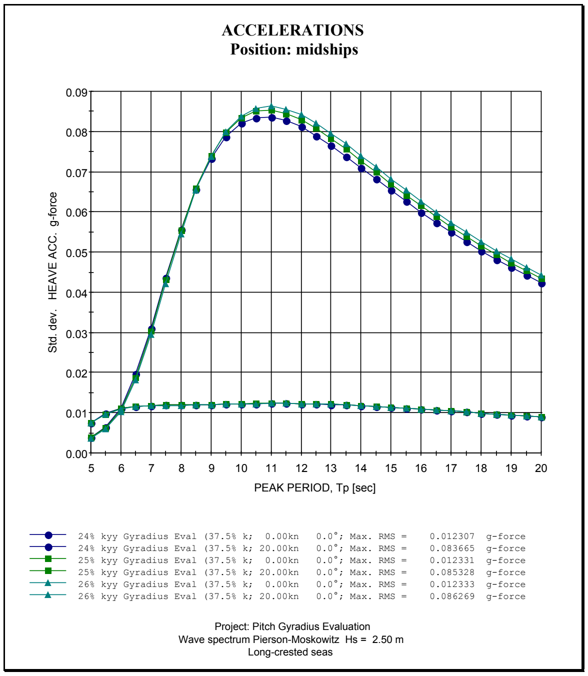


Figure 9: RMS Heave Acceleration @ Midships vs. Peak Period

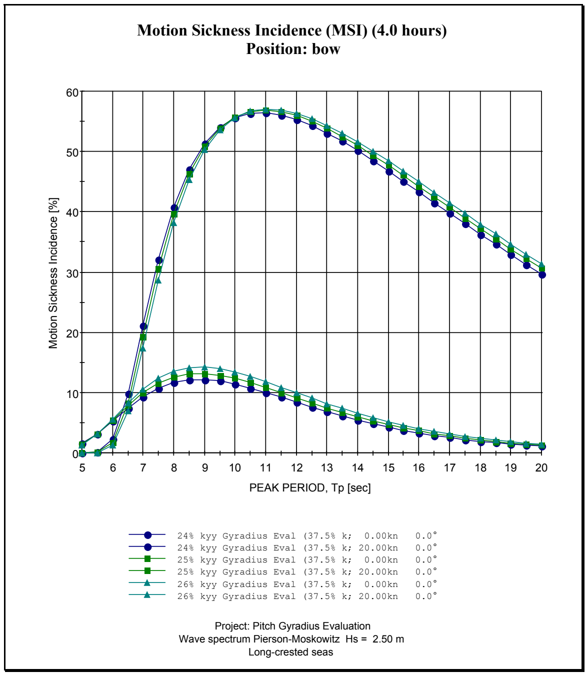


Figure 10: Motion Sickness Incidence @ Bow vs. Peak Period

Roll Gyradius, k_{xx}/BWL	Max RMS Roll (deg)	Variation from 37.5% value (deg)	Variation from 37.5% value (%)
35%	7.40	-0.04	-0.54%
37.5%	7.44	-	-
40.0%	7.49	0.05	0.67%

Table 2: Maximum RMS Roll Values for a Range of Gyradii

Pitch Gyradius, kyy/LWL	Max RMS Pitch (deg)	Variation from 25% value (deg)	Variation from 25% value (%)
24%	0.896	-0.034	-3.66%
25%	0.930	-	-
26%	0.968	0.038	4.09%

Table 3: Maximum RMS Pitch Values for a Range of Gyradii, 0 knots

Pitch Gyradius, k_{yy}/LWL	Max RMS Pitch (deg)	Variation from 25% value (deg)	Variation from 25% value (%)
24%	1.183	-0.006	-0.50%
25%	1.189	-	-
26%	1.192	0.003	0.25%

Table 4: Maximum RMS Pitch Values for a Range of Gyradii, 20 knots

Tables 2, 3, and 4 show the variations of maximum RMS (which for responses that have a zero mean is equivalent to standard deviation) roll and pitch response magnitudes for the low and high values of gyradius, compared to the middle value. While it can be seen that the variations in roll, and in pitch at 20 knots are relatively small, it's difficult to generalize the results over a range of vessel types and sizes. One important point to note is that because of the formulation of the OI, where exceeding a specified value of a particular motion limit changes the operability value $O(V_S,\psi)$ from 1 to 0, even a very small increase can mean the difference between the vessel being considered to be able to complete a mission or not. For example, if an SSA roll limit of 8 degrees is specified, a vessel with an SSA roll angle of 7.9 degrees will be considered to be fully operable, whereas a vessel with an SSA roll angle of 8.1 degrees will be considered inoperable. While the peak RMS roll response magnitude does not vary greatly in Figure 6, the peak shifts. A comparison of values at a constant period will yield much larger variation, which could significantly impact the OI. The shift in the period of the peak values of the other responses studied was comparatively small.

The vessel's moment of inertia (and resulting gyradius) is just one of many factors that enter into the prediction of its motions in a seaway, and an evaluation of its performance with regard to the mission. Small changes in gyradius would seem (at least for the examples presented here) to lead to small changes in motions, and may become insignificant factors when compared to other factors such as active roll damping, variations in limiting criteria, uncertainty in operational profile, and estimates of sea conditions. For a unique ship design, or a design where seakeeping performance is critical and performance seems to be near the boundary of acceptability, detailed computation of the MOI could be important. For a design that appears to be well within the seakeeping limiting criteria limits, when analyzed using MOI estimates based on similar existing designs, there may not be as much value in the detailed computation of MOI.

Methods of Calculating Moment of Inertia

Accurate Calculation of MOI

Accurate calculation of an object's MOI requires (following the mathematical definition given earlier) integration of the mass with respect to an axis. To be able to do this one needs to develop a geometric definition of the object as well as know its density and the distribution of that density.

In practice, one would need to model the object accurately using CAD software to be able to calculate an accurate moment of inertia.

Approximations to the MOI

Since accurate calculation of MOI for a vessel is not practical during design, most weight engineers use approximations to calculate the moment of inertia.

The most common way of approximating the MOI is by looking at individual items of the total vessel and combining the use of the Parallel Axis Theorem (often referred to as "Steiner's Theorem") with the approximation of an object's geometry to a form whose MOI is known.

When one is in possession of a weight database giving the weight and location of individual items of a vessel, each item's contribution to the total MOI for the vessel can be found by squaring the distance from the local center of gravity of the item to the global center of gravity of the complete vessel and multiplying this by the item's mass. This is the simplest form of approximation, and the self inertia (I_o) of the item is neglected in this case. Still, if the number of items is significant, the error in neglecting the self inertia might be regarded as small.

If the information about shape and extents in three dimensions for an item is known, adding an approximate value of self inertia (I_o) may be done by simplifying the item to an object where calculation of the MOI can be done easily. This is especially useful for large objects where the self inertia may have substantial impact on the total MOI of the vessel.

Table of Simplified Objects for MOI Calculations

The table below[8] shows a selection of objects that might be used for a simplified calculation of self inertia.

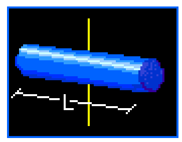
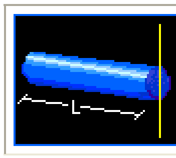
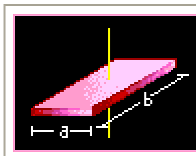
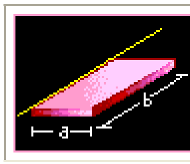
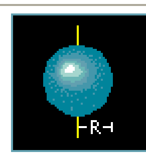
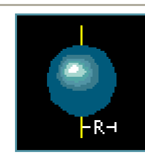
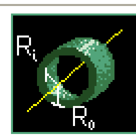
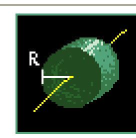
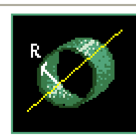
slender rod:	axis through center	$I = \frac{1}{12}ML^2$		axis through end	$I = \frac{1}{3}ML^2$	
rectangular plane:	axis through center	$I = \frac{1}{12}M(a^2 + b^2)$		axis along edge	$I = \frac{1}{3}Ma^2$	
sphere	thin-walled hollow	$I = \frac{2}{3}MR^2$		solid	$I = \frac{2}{5}MR^2$	
cylinder	hollow	$I = \frac{1}{2}M(R_i^2 + R_o^2)$		solid	$I = \frac{1}{2}MR^2$	
	thin-walled hollow	$I = MR^2$				

Figure 11: Table of MOI formulas for simple objects

Parallel Axis Theorem (Steiner's Theorem)

In physics, the parallel axis theorem can be used to determine the moment of inertia of a rigid body about any axis, given the moment of inertia of the object about the parallel axis through the object's center of mass and the perpendicular distance between the axes[9].

Let:

x be an axis through an object's center of mass
I_x denote the moment of inertia of the object about this axis
m denote the object's mass and *d* the perpendicular distance between the two axes.

Then the moment of inertia about the new axis *z* is given by:

$$I_z = I_x + md^2$$

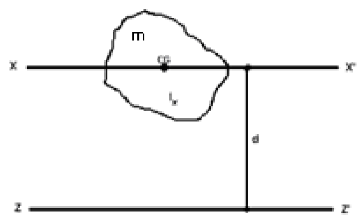


Figure 12: Parallel axis theorem

Challenges and Problems Related to the MOI Calculation

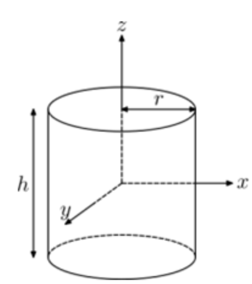
Due to the nature of MOI calculations, most weight engineers rely on approximation of the MOI. But the approximation raises several questions:

- For which and how many items do I need to consider the self inertia (I_o)?
- Which simplified object should be selected to represent the actual object?
- How do I account for density variations in the object?
- Do I have the information needed to calculate the simplified object?

The Accuracy of a Simplified Object

Manufacturers of MOI measurement tools claim that "Typical errors in calculated MOI can range to over 30% due to simplifying the part shape, or making assumptions about average density."[10]

By looking at a small example, we may see that this claim is not unreasonable. Consider an object approximated to the shape of a solid cylinder. The MOI around the z axis is:



$$I_{z-estimate} = \frac{mr^2}{2}$$

Figure 13: MOI of a solid cylinder

If this was an approximation and the actual object was a hollow cylinder, then the actual and correct MOI for the object would be:

$$I_{z\text{-actual}} = \frac{m\left(r_1^{\,2} + r_2^{\,2}\right)}{2}$$

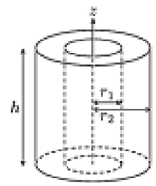


Figure 14: MOI of a hollow cylinder

If we let $r_2=2r_1$, then the relationship between $I_{z\text{-estimate}}$ and $I_{z\text{-actual}}$ would be: $\dfrac{\dfrac{mr_2^{\,2}}{2}}{\dfrac{m\left(\dfrac{r_2^{\,2}}{4}+r_2^{\,2}\right)}{2}} = \dfrac{4}{5}$, implying that the estimate is 20 % lower than the real value. In addition to the approximation of the item shape, in reality you would often also approximate density distribution and position which may further increase the error.

Current Practice Dealing with These Challenges

Examinations to help deal with these challenges within the marine industry have been carried out and a couple of examples of such examinations have been presented by Cimino and Redmond, and Hansch[11].

The first of these papers, "Naval Ships' Weight Moment of Inertia," examines the relation between transference inertia (parallel axis theorem method) and self inertia (I_o) to develop guidelines for early phase inertia estimation - "rules of thumb." It examines naval vessels to establish ways of determining which and how many of the items in a weight database need a simplified geometry defined for self inertia calculation to obtain an acceptable value of the total inertia.

The second paper, "Weight Distribution Method of Determining Gyradii of Ships," proposes a new method of calculating the radii of gyrations through numerical integration of the weight distribution. This paper shows that it is possible to determine the weight moment of inertia by calculating the weight distribution along the principal axes and then calculating the MOI by integrating the distribution.

Uncertainty and Confidence in Results

Although the SAWE papers referred to above provide good insight and recommendations regarding calculation of MOI for vessels, one may suggest that they do not provide an accurate way of determining the uncertainty and confidence of the results they provide. Further it may be argued that conditions and constraints under which their methods will provide adequate results are not clearly defined, such as type of vessel, number of items in weight database, etc.

In Cimino's and Redmond's paper, the uncertainty of the MOI value for simplified part shapes is not considered and the example with the cylinder above shows that this might imply significant errors in the self inertia calculation.

In Hansch's paper, the uncertainty of the method lies in that the weight distribution, which is the basis of the integration, is an approximated curve in itself.

The method in this paper does not eliminate the error of an MOI estimation, but it quantifies it and tells you where to work to reduce it.

Proposed MOI Calculation Including Error Range Quantification

To solve the problem of the lack of accuracy quantification of the MOI calculation, the method described in this paper will try to propose a calculation that shifts the focus from trying to find a specific value for the MOI to finding the absolute minimum and maximum MOI value a vessel can possibly have, and thus establish an MOI value with an exact error range.

Considering Accuracy of the Parallel Axis Theorem (Steiner's Theorem)

As shown above, Steiner's theorem defines a single object's MOI contribution to a larger, composed object's MOI by squaring the distance times mass and adding the self inertia for each individual item of the complete vessel. The parallel axis theorem can be applied to all weight items on a vessel in the following way:

$$I_{total} = I_t + I_o \quad \text{and} \quad I_t = \sum_{i=1}^{n} m_i r_i^2$$

Where I_t = transference inertia

I_o = self inertia

r = distance from global center of gravity to local center of gravity

m = mass of item

As we would normally know ***m*** and ***r***, $\boldsymbol{I_t}$ may be calculated accurately (at least to the extent that ***r*** and ***m*** are accurate) and the inaccuracy lies in the calculation of the self inertia $\boldsymbol{I_o}$.

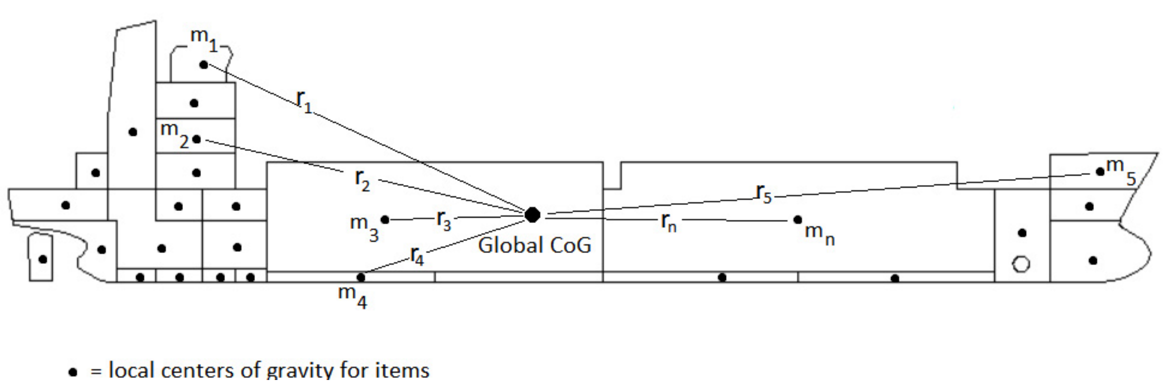


Figure 15: Parallel axis theorem applied on a vessel

Minimum Value of Self Inertia (I_o)

The minimum value the self inertia I_o can obtain is 0. This follows from the mathematical definition of MOI and from the extreme situation of concentrating all mass of the object in a single point located at the local center of gravity for the object.

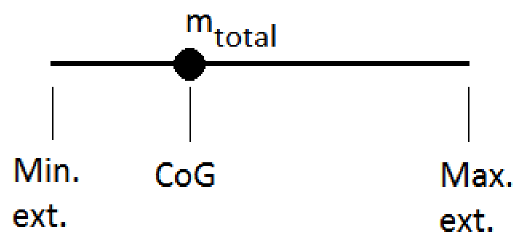


Figure 16: All mass located at the point of center of gravity (CoG)

Thus, from $I_o = \int (a^2 + b^2) dm$ we get $I_{o_{min}} = 0$ since ***a*** and ***b*** are 0 when ***dm*** > 0; and ***dm*** = 0 when ***a*** and ***b*** are > 0.

Maximum Value of the Self Inertia (I_o)

The maximum value the self inertia I_o can possibly have would be concentrating the mass at the end points of the extension of the item.

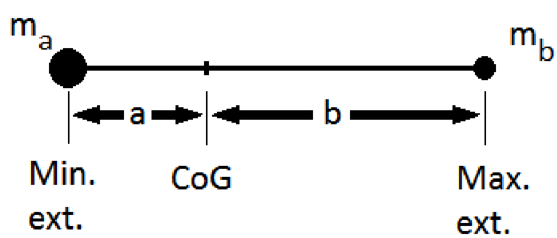


Figure 17: Distributing the mass to points at the extension ends will maximize the moment of inertia

From the mathematical definition of the MOI we have (1): $I_o = \int (a^2 + b^2) dm$

For the case above, this means (2): $I_{o_{max}} = m_a a^2 + m_b b^2$

We have the following (3): $m_{total} = m_a + m_b$

To preserve the center of gravity, we know that (4): $m_a a = m_b b$

Combining (3) and (4) gives (5): $m_a = \dfrac{m_{total}}{\left(1 + \dfrac{a}{b}\right)}$

And when knowing the m_{total} and distances a and b, we can calculate m_a from (5) and m_b from (3) after obtaining m_a.

Approximating MOI and Finding the Accurate Error Range

Once the minimum and maximum values of an object are found, the approximation of the self inertia would be the average value of the maximum and minimum value (remembering that $I_{O_{min}} = 0$):

$$I_{O_{average}} = I_{O_{max\,error}} = \frac{(I_{O_{max}} - I_{O_{min}})}{2} = \frac{I_{O_{max}}}{2}$$

And thus the error range would be given by:

$$I_{O_{error\,range}} = I_{O_{average}} \pm \frac{I_{O_{max}}}{2}$$

The approximation of the MOI for the complete vessel would be found simply by adding the MOI found for each item and the total error range for the complete vessel would be found in similar way by adding the error ranges.

$$Total\,I_{O_{maxerror}} = \sum \left(\frac{I_{O_{max}}}{2} \right)$$

Example Calculation

Let's consider some sample data. The below is a snippet of data from a weight database of a vessel (VCG, LCG, TCG are referenced from the origin, not the global center of gravity):

Weight [kg]	VCG [m]	LCG [m]	TCG [m]	VCGmin [m]	VCGmax [m]	LCGmin [m]	LCGmax [m]	TCGmin [m]	TCGmax [m]
293	1.73	102.90	-0.95	0.38	2.14	101.06	107.02	-1.10	-0.69
1144	5.64	10.60	-0.09	4.70	5.79	9.83	10.91	-0.18	0.58
2994	5.37	60.33	0.47	4.82	7.14	55.64	62.70	-0.13	1.04
369	7.90	89.55	0.12	7.66	8.73	84.88	94.29	-0.49	0.73
96	0.00	0.00	0.00	-2.08	0.21	-2.86	0.58	-0.27	0.10
752	8.50	1.50	-0.60	7.92	10.58	-1.93	5.00	-1.57	-0.25
3523	5.02	41.27	0.40	3.42	6.85	36.41	44.12	-0.49	0.99
8579	9.69	40.00	0.00	7.87	10.84	35.57	44.25	-0.37	0.66
68	10.30	33.00	0.00	9.56	12.14	32.70	35.69	-0.34	0.29
211	10.16	3.88	0.00	9.36	10.61	0.63	7.59	-0.72	0.91
121	15.93	92.82	-6.62	15.36	16.22	89.05	95.58	-7.36	-5.62
2325	4.85	61.30	0.63	4.40	6.89	59.02	62.21	-0.21	0.71
887	7.10	90.60	0.15	6.08	8.94	86.19	92.41	-0.02	0.71
294	9.83	26.70	-9.73	8.98	9.99	22.25	26.94	-10.42	-8.90

Table 5: Weight database sample

Using the proposed method would give the following results for yaw, pitch and roll, where:

m_{total}	=	Total item weight
r	=	Distance between global CoG and item (local) CoG
a	=	Distance from item (local) CoG to item's start point
b	=	Distance from item (local) CoG to item's end point
m_a	=	Part weight transposed to item's start point
m_b	=	Part weight transposed to item's end point
I_t	=	Transference inertia ($m_{total} * r^2$)
I_o	=	Self inertia ($\frac{m_a a^2 + m_b b^2}{2}$)
I_{total}	=	Total inertia ($I_t + I_o$)
I_{error}	=	Max total inertia error according to method ($100 * \frac{I_o}{I_t}$)

m_{total}	r	a	b	m_a	m_b	I_t	I_o	$I_{z\text{ - }total}$	$I_{z\text{ - }error}$
293	52.4	2.19	1.36	112.4	180.8	805350	438	805788	0.1 %
1144	39.9	0.71	4.52	988.1	155.7	1820963	1838	1822801	0.1 %
2994	9.8	1.69	1.17	1224.6	1769.6	289858	2961	292820	1.0 %
369	39.1	2.31	2.54	193.1	176.0	562859	1082	563940	0.2 %
96	50.5	1.06	0.70	38.4	57.7	245028	36	245064	0.0 %
752	49.0	5.01	4.98	375.1	376.9	1805870	9381	1815251	0.5 %
3523	9.2	1.19	0.46	987.5	2535.3	300745	970	301715	0.3 %
8579	10.5	1.72	4.25	6108.1	2471.0	945869	31391	977260	3.2 %
68	17.5	4.74	0.66	8.3	59.2	20672	106	20778	0.5 %
211	46.6	0.65	4.06	181.4	29.1	457715	279	457994	0.1 %
121	42.8	4.37	3.69	55.3	65.4	221541	972	222514	0.4 %
2325	10.8	0.58	0.98	1465.8	858.7	271903	658	272561	0.2 %
887	40.1	1.72	1.74	446.2	440.7	1426105	1326	1427431	0.1 %
294	25.7	2.32	3.67	180.4	113.9	194824	1253	196077	0.6 %
21655						**9369303**	**52691**	**9421994**	**0.6%**

Table 6: Yaw calculations

m_{total}	r	a	b	m_a	m_b	I_t	I_o	$I_{y\text{ - total}}$	$I_{y\text{ - error}}$
293	53.0	2.51	1.96	128.7	164.5	823355	723	824079	0.1 %
1144	40.1	0.57	4.67	1019.6	124.2	1839196	1518	1840714	0.1 %
2994	10.7	2.21	1.47	1194.9	1799.3	343668	4849	348517	1.4 %
369	39.1	2.62	2.67	186.5	182.6	563961	1289	565251	0.2 %
96	51.4	1.86	1.26	38.8	57.3	253938	113	254051	0.0 %
752	49.0	5.18	5.01	369.6	382.4	1806512	9767	1816280	0.5 %
3523	10.3	1.70	0.47	759.0	2763.8	375213	1397	376610	0.4 %
8579	10.5	1.86	4.53	6086.7	2492.5	945874	36078	981952	3.7 %
68	17.5	4.89	1.53	16.1	51.4	20702	253	20955	1.2 %
211	46.6	0.45	4.22	190.3	20.2	457773	199	457972	0.0 %
121	42.8	4.43	3.80	55.7	65.0	220962	1015	221977	0.5 %
2325	11.8	1.14	1.49	1315.5	1009.1	324246	1973	326220	0.6 %
887	40.2	1.69	1.74	450.2	436.7	1431773	1305	1433078	0.1 %
294	23.8	2.50	3.90	179.4	114.9	166715	1435	168150	0.9 %
21655						**9573890**	**61916**	**9635806**	**0.6%**

Table 7: Pitch calculations

m_{total}	r	a	b	m_a	m_b	I_t	I_o	$I_{x\text{ - total}}$	$I_{x\text{ - error}}$
293	8.0	1.80	1.43	129.8	163.4	18592	377	18969	2.0 %
1144	4.0	0.88	1.19	656.6	487.2	18277	601	18879	3.2 %
2994	4.3	1.91	1.44	1287.2	1707.1	54866	4116	58983	7.0 %
369	1.7	1.23	1.67	212.8	156.3	1107	377	1484	25.4 %
96	9.6	1.70	1.07	37.2	58.9	8910	87	8998	1.0 %
752	1.3	1.66	0.88	260.0	492.0	1278	550	1828	30.1 %
3523	4.6	1.75	0.07	143.4	3379.4	75331	229	75560	0.3 %
8579	0.1	0.70	1.80	6181.7	2397.4	47	5382	5429	99.1 %
68	0.7	1.22	1.50	37.2	30.3	30	62	92	67.0 %
211	0.5	0.73	1.39	138.2	72.3	59	107	165	64.6 %
121	9.2	0.85	0.92	62.7	58.0	10160	47	10207	0.5 %
2325	4.8	1.01	1.71	1460.2	864.3	53881	2015	55895	3.6 %
887	2.5	0.33	0.76	616.2	270.7	5686	113	5798	1.9 %
294	9.8	1.44	1.87	166.2	128.1	28132	396	28528	1.4 %
21655						**276357**	**14459**	**290816**	**5.0%**

Table 8: Roll calculations

Utilizing the Error Range to Improve the Total MOI Calculation

Finding the MOI error range (the sum of the I_o columns in Tables 6, 7, and 8) is helpful in at least two ways. First, it will enable you to say more about the accuracy of the calculations where the MOI is used for input, such as seakeeping, added resistance, strength and fatigue calculations. Secondly, it will allow you to find the items contributing most to the total MOI inaccuracy and work further on these items to improve the accuracy on these items.

Finding and Improving the Most Inaccurate Items

The most inaccurate items are found simply by sorting your items with regard to the self inertia max error range ($\frac{I_{o_{max}}}{2}$) calculated. This will automatically give you the list of which items to work on. Note that the percentage error can be misleading and shouldn't be used to determine where to focus your efforts. Two identical items, one near the global center of gravity, and one far away, will have identical I_o values but could have vastly different percentage error values. The accuracy of the total inertia is equally influenced by each of these items, so you must focus on the magnitude of I_o, rather than its percentage of the total inertia contributed by an item.

Once these items are identified, you can improve the accuracy for these items either by

- Calculating the self inertia accurately by making an exact model in CAD and calculate MOI from integration on the geometry; or
- Dividing the item into several sub-items, thus reducing the total inaccuracy

Note that using a simplification of an object is not mentioned here as an alternative to improving the overall MOI for the very reason that we would not be able to say anything about the error range for this item.

Getting the self inertia from a CAD system would be an accurate, but a potentially labor intensive way of improving the MOI as the item would have to be modeled to a very detailed level. Dividing the item into sub items is less work intensive, but still could provide a surprisingly large improvement as will be shown below.

The Power of Dividing an Item into Sub Items

Consider a "vessel" built as a solid, homogenous box with the following dimensions (metric units applied, but quantities are irrelevant here):

- L (length) = 80 m
- B (beam) = 10 m
- H (height) = 10 m
- M (mass) = 100 tonnes

From literature we'll find that the MOI in pitch direction for this object can be determined from the formula:

$I_{yy} = k(L^2 + H^2)$, where $k = \frac{1}{12}$ for a homogenous box.

Given the parameters above, I_{yy} = 54167 tonnes-m^2.

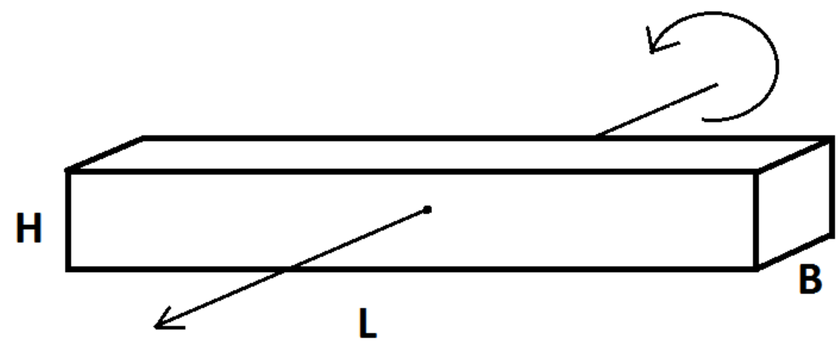


Figure 18: The homogenous box "vessel"

It may be shown that when dividing this box into sub boxes and using the parallel axis theorem, combined with the self inertia of the sub boxes, the part of transference inertia (I_t) and the part of self inertia from the sub boxes (I_o) may be found from the following formulas:

$$I_o = kM\left[\left(\frac{L}{n}\right)^2 + H^2\right]$$

and

$$I_t = 2M\sum_{i=1}^{n/2}\left(\frac{Li}{n} - \frac{L}{2n}\right)^2$$

Where $k = \frac{1}{12}$ for a homogenous box

n = number of sub boxes

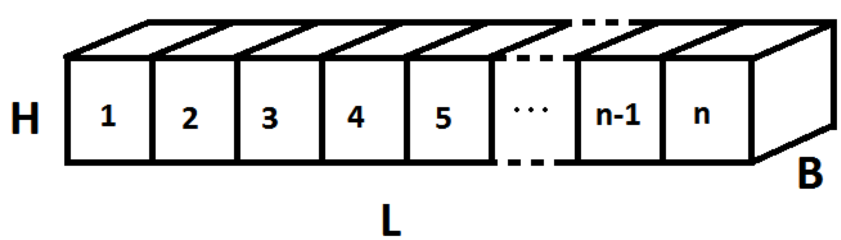


Figure 19: The homogenous box "vessel" divided into sub boxes

Using these formulas, we can find the following relations between number of sub boxes, the transference inertia and self inertia, and thus the error from neglecting the self inertia.

Number of sub boxes	MOI from transference	Self Inertia	Error from neglecting self inertia
1	0	54167	100.0 %
2	40000	14167	26.2 %
4	50000	4167	7.7 %
8	52500	1667	3.1 %
16	53125	1042	1.9 %

Figure 20: Error from neglecting self inertia as a function of segments

From this table we see that when splitting a box into sub boxes, the error from neglecting the self inertia rapidly becomes insignificant and proves that this can be a valid way of improving the accuracy of an object's MOI compared to that of establishing an approximate shape of an object. It should be noted that the error for neglecting self inertia will vary depending upon the *k* factor of the object.

Advantages and Disadvantages of the Method

Advantages

The advantages of this method may be summarized as follows:

- The method will provide an exact error range; you will know the minimum and maximum MOI the object can possibly have.
- You can find the items that contribute most to the overall error range and target these items to improve your estimate, making sure you work no more and no less than needed to fulfill the accuracy needed.
- You do not need to define and position simplified geometric objects to deal with the self inertia of individual items and the uncertainty that these simplifications imply.

Disadvantages

The biggest disadvantage of the method is that it requires the extensions in all three directions for the items involved in the calculation. In case of extracting information from a CAD system this might be as easy as defining the output to include these quantities, but in case of manual input, this might represent a significant amount of work.

Conclusion and Summary

If you are able to obtain extensions for the weight items in the database with a reasonable amount of work, the presented method gives a fast way of obtaining the MOI, including the error range of the answer. If the confidence level is too low, the method will pinpoint the items that will improve on the overall accuracy and allow you to spend the resources needed, neither more nor less, to obtain the answer with an accepted uncertainty.

Perhaps the method is particularly useful in situations where projects fall outside the scope of previously proposed methods in terms of number of items in the database and/or the vessel type.

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